# The Quantum Mechanics of Supersymmetry

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Bachelor of Science

by

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# *I. Introduction*

Most, if not all, physicists are familiar with the idea of a grand unified theory (GUT). The general idea behind a GUT is to have a single framework that succinctly describes all of the fundamental interactions of nature: gravity, electromagnetism (EM), strong nuclear, weak nuclear. The current dominating model of physics is the Standard Model (SM), which unifies the EM, strong, and weak interactions with a reasonable level of success, but has no mechanism to explain gravity. It is precisely this problem that modern GUTs seek to resolve. Unfortunately, there are about as many GUTs as there are physicists in the field. That being said, there appear to be some recurring elements in many such theories. One of the more intriguing aspects to many GUTs is the idea of supersymmetry (SUSY). The fundamental idea behind SUSY is that for every elementary particle, there exists a "superpartner" particle that differs in spin by a half unit. As a reminder, all elementary particles possess an intrinsic angular momentum characterized by their spin number. Particles with integer spin (s = 0, 1, 2, …) are called bosons and particles with half-integer spin (s = 1/2, 3/2, 5/2, …) are called fermions. Although they will not be discussed in detail here, bosons and fermions possess many unique and interesting properties due to their spins and are experimentally easy to differentiate. SUSY is particularly interesting then because it states that every boson has a fermion superpartner and vice versa. In more technical terms, one would say that SUSY relates bosonic and fermionic degrees of freedom.

SUSY was first used in string theory to unify the bosonic and fermionic sectors but turned out to have many other interesting properties when other groups started to apply it to their theories. Of particular importance was the generalization of gravity to incorporate SUSY. In such theories where gravity is invariant under SUSY, i.e. there is some gravitational parameter that

remains constant under supersymmetric transformations, a supergravity theory results. These theories are among the leading candidates for physics beyond the SM and thus represent some of the first steps towards a GUT.

In the following, the mechanisms and algebras typically associated with supersymmetry will be applied to rudimentary quantum mechanics. The level of the paper will be kept to that of a student who has taken, or will soon take, an undergraduate quantum mechanics course. Since this analysis will be kept at such a level, no direct conclusions will be drawn on the nature of supersymmetry as it applies to quantum mechanics or vice versa. Rather, it will serve as an intellectual exercise that will shed some insight into the underlying mathematical structure of supersymmetry. Some familiarity with undergraduate quantum mechanics is required so a brief summary/review is provided here. If the reader feels sufficiently comfortable in this regard, then the remainder of this introduction can be skipped without loss of continuity.

One of the most basic ideas involved in both quantum mechanics and classical mechanics is the Hamiltonian. In its simplest interpretation, the Hamiltonian is the total energy of a given system, i.e., it is the sum of a system's potential and kinetic energies.

$$\mathcal{H} = T + V \tag{1}$$

In equation 1, T is the kinetic energy and V is the potential energy. In a classical system, both T and V would be calculated based on the dynamics of the system. In quantum mechanics, both T and V must be calculated using a branch of mathematics called operator theory. A proper treatment of operator theory as it applies to quantum mechanics can be found in any respectable undergraduate text (e.g. David Griffiths' Introduction to Quantum Mechanics) and the reader is encouraged to gain a deeper understanding in such a text. Concisely speaking, operators are used

as a sort of query applied to a system. For example, in order to calculate the kinetic energy of the system, one would apply the kinetic energy operator, notated by $\hat{T}$ (read "t-hat"). The object that is operated *on* in quantum mechanics is called the wave function, denoted by Ψ or ψ, and is a function of both position and time. Ψ typically represents a particle, but it is important to note that Ψ also contains information about the environment the particle exists in. For example, a particle in free space will have a different wave function than if that same particle were in a confined box. This is a difficult concept to grasp because there is no macroscopic analog to this scenario. If one wanted to find the kinetic energy of a particle-environment system, $\hat{T}$ would be applied to ψ. For the sake of conciseness, $\hat{T}$ is given below without explanation:

$$\hat{T} = \frac{\hat{p}^2}{2m}. \tag{2}$$

Here, $\hat{p}$ is the momentum operator. (Note: the "squared" notation implies that the operator is applied twice.) $\hat{p}$ is given in equation 3:

$$\hat{p} = -i\hbar\nabla. \tag{3}$$

Notice the similarity to classical mechanics where the kinetic energy T of an object with momentum p is given by the same equation without the hats. Naturally, one would reason that applying the momentum operator to a wave function should return its momentum. But this is only true if ψ satisfies a certain relationship with the momentum operator. That is, ψ must be a 'momentum eigenstate'. Mathematically, this just means that Ψ and $\hat{p}$ satisfy a type of equation

called an 'eigenvalue equation'. These types of equations play a fundamental role in quantum mechanics and should be well understood by anyone studying this field. Details on eigenvalue equations as they pertain to quantum mechanics can be found in any respectable undergraduate text and will not be discussed in this paper. The potential energy operator, $\hat{V}$, is much more straightforward. $\hat{V}$ acting on ψ is simply ψ multiplied by the potential field V(**r**) where **r** is the position vector. The Hamiltonian operator, $\hat{H}$, can now be constructed:

$$\hat{H} = \hat{T} + \hat{V}. \tag{4}$$

Applying $\hat{H}$ to the wave function returns the total energy of the system, provided that ψ is an energy eigenstate. If the Hamiltonian is written out in its complete form, the result is the familiar time-independent Schr□dinger equation (SE)

$$\hat{H}\psi = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi. \tag{5}$$

Note that the first term in the middle expression is equation 2 with the momentum operator expanded out. It is important to recognize that equation 5 has 3 unknowns: the wave function ψ, the energy E, and the potential V. In ordinary algebra, one would be required to know 2 of the variables to determine the third. However in quantum mechanics, there are several boundary conditions that must be met and even forced upon the solutions to the time-independent Schr□dinger equation. The result is that, for many attractive potentials, the solutions become quantized. That is, given some potential V, only certain wave functions can exist within that potential. Furthermore, each wave function solution has a specific energy associated with it. The

upshot is that not all energies are allowed in every potential; only those energies that are represented by a wave function solution to the SE given that specific V are permitted. This effectively reduced the number of unknowns in the SE to 2. Once calculated, the collection of energy solutions is known as the spectra of solutions for a particular potential. In each spectrum, there is a lowest energy solution called the ground state which is denoted by $E_0$. The associated wave function is called the ground state wave function and is denoted by $\psi_0$. Typically, the potential is given and the solution spectra must be calculated. It is possible to reconstruct the potential given a solution spectrum and the associated wave functions, but this is a much more difficult problem and will not be discussed at length in the following analysis. Provided below is a quick example that will be relevant to later discussion.

The canonical example is the one dimensional infinite square well. The potential is plotted in figure 1 below.

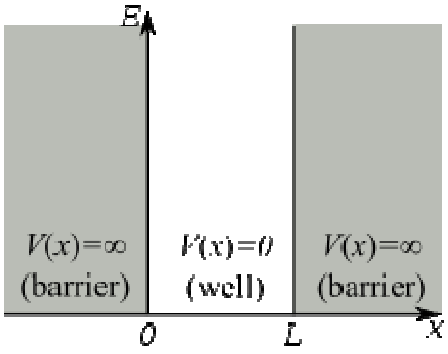



**Figure 1: Infinite Square Well**

Mathematically, this potential is given by equation 6 below:

$$V(x)=\begin{cases}0, & 0\le x\le L\\ \infty, & x<0 \quad or \quad x>L\end{cases}. \tag{6}$$

Here, L is the length of the well. Now place a particle of mass m within the well. The SE within the well is then given by

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi + 0 = E\psi. \tag{7}$$

This is a differential equation that is relatively easy to solve. The solution to this problem can be found in Griffiths or any text on differential equations. The solutions are presented below without proof:

$$\psi_n = \sqrt{\frac{2}{L}}\sin\left((n+1)\frac{\pi x}{L}\right), \qquad 0 \le x \le L \tag{8}$$

$$E_n = (n+1)^2\frac{\hbar^2\pi^2}{2mL^2}\ . \tag{9}$$

Note that n ≥ 0. Equations 8 and 9 represent complete solution spectra for the given infinite well. These wave functions and energies will be built upon later in the analysis.

# *II. Analysis*

Consider the time-independent Schr□dinger equation as shown in equation 5. To simplify the analysis, assume the ground state energy is 0 and that the wave function is dependent only on the x-direction:

$$\hat{H}_1\psi_0(x) = -\frac{\hbar^2}{2m}\psi_0''(x) + V_1(x)\psi_0(x) = 0. \tag{10}$$

The 1 subscripts on the Hamiltonian and potential terms will make more sense later. Solving for the potential gives equation 7:

$$V(x) = \frac{\hbar^2}{2m} \frac{\psi_0''(x)}{\psi_0(x)}. \tag{11}$$

This is an important result in general because it allows one to reconstruct the potential by only knowing the ground state wave function. This facet of equation 11 will not come in future use in the analysis, but is an important point nonetheless. Now return to equation 10. The typical next step in a quantum mechanical treatment of supersymmetry would be to factorize the Hamiltonian into two operators in the following fashion:

$$\hat{H}_1 = A^\dagger A = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_1(x) \tag{12}$$

$$A = \frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + W(x) \tag{13}$$

$$A^\dagger = \frac{-\hbar}{\sqrt{2m}} \frac{d}{dx} + W(x). \tag{14}$$

Notice that there are no hats on the A operators, but they behave just as the previously mentioned operators do. For simplicity, the hats will be removed from all operators for the remainder of the paper. To see that the Hamiltonian can in fact be constructed this way, one can apply the two operators in succession. Applying A to $\psi_0$, one gets

$$A\psi_0 = \frac{\hbar}{\sqrt{2m}}\psi_0' + W\psi_0. \tag{15}$$

Now applying $A^\dagger$,

$$A^\dagger A\psi_0 = -\frac{\hbar}{\sqrt{2m}}\frac{d}{dx}\left[\frac{\hbar}{\sqrt{2m}}\psi_0' + W\psi_0\right] + W\left(\frac{\hbar}{\sqrt{2m}}\psi_0' + W\psi_0\right)$$

$$A^\dagger A\psi_0 = -\frac{\hbar}{\sqrt{2m}}\frac{d}{dx}\left[\frac{\hbar}{\sqrt{2m}}\psi_0' + W\psi_0\right] + W\frac{\hbar}{\sqrt{2m}}\psi_0' + W^2\psi_0$$

$$A^\dagger A\psi_0 = -\frac{\hbar}{\sqrt{2m}}\left(\frac{\hbar}{\sqrt{2m}}\psi_0'' + W\psi_0' + W'\psi_0\right) + W\frac{\hbar}{\sqrt{2m}}\psi_0' + W^2\psi_0$$

$$A^\dagger A\psi_0 = -\frac{\hbar^2}{2m}\psi_0'' - \frac{\hbar}{\sqrt{2m}}W'\psi_0 + W^2\psi_0$$

$$A^\dagger A\psi_0 = -\frac{\hbar^2}{2m}\psi_0'' + \left(W^2 - \frac{\hbar}{\sqrt{2m}}W'\right)\psi_0. \tag{16}$$

The last line in equation 16 can now be compared directly with equation 10. One then finds that under this formulation of the Hamiltonian, the potential V can be expressed in terms of W.

$$V_1(x) = W^2(x) - \frac{\hbar}{\sqrt{2m}}W'(x) \tag{17}$$

The W potential is referred to as the superpotential and plays an integral role in SUSY QM. To truly appreciate its importance, return to equation 15. Notice that if equation 15 is set to 0, the solution function for equation 16 will also be a solution to equation 10, since $A^\dagger(A\psi_0)=A^\dagger(0)=0$. Doing so yields

$$\frac{\hbar}{\sqrt{2m}}\psi_0' + W(x)\psi_0 = 0 \tag{18}$$

$$W(x) = -\frac{\hbar}{\sqrt{2m}}\frac{\psi_0'}{\psi_0}. \tag{19}$$

The superpotential can now be calculated in terms of the ground state wave function, but W has yet to show any usefulness other than its ability to recast the Hamiltonian. Consider again the A operators in equations 12, 13, and 14. Now consider what would happen if the operators in equation 12 switched order:

$$H_2 = AA^\dagger = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_2(x). \tag{20}$$

The derivation of $V_2(x)$ will not be shown explicitly because it is identical to that of $V_1(x)$ which is shown in the steps preceding equation 16.

$$V_2(x) = W^2(x) + \frac{\hbar}{\sqrt{2m}}W'(x) \tag{21}$$

$V_1$ and $V_2$ are called supersymmetric partner potentials and they are related through the superpotential W, which is calculated via equation 19. But in order to calculate W through equation 19, the ground state wave function must be known for $H_1$, notated from this point forward as $\psi_0^{(1)}$. In principle, W can also be calculated by solving $A^\dagger\psi_0^{(2)}=0$ in which case only the ground state of $H_2$ must be known. The expression for W would be identical to that in equation 19 but without the negative sign and with the ground state wave function for $H_2$. Note that the wave functions in equation 19 are from $H_1$. With all of the calculations presented here, it would be reasonable for one to assume that the wave functions of $H_1$ and $H_2$ are related in some

way. To confirm this, setup the SE as in equation 5 but specify which Hamiltonian is being used. To preserve generality, the order of the wave function, n, will not be specified:

$$H_1\psi_n^{(1)} = A^\dagger A\,\psi_n^{(1)} = E_n^{(1)}\psi_n^{(1)} \tag{22}$$

$$H_2\psi_m^{(2)} = AA^\dagger\psi_m^{(2)} = E_m^{(2)}\psi_m^{(2)}. \tag{23}$$

Consider now if the operator A acted upon the middle expression of equation 22. Knowing that $H_2=AA^\dagger$,

$$A(A^\dagger A\,\psi_n^{(1)}) = H_2(A\,\psi_n^{(1)}). \tag{24}$$

Additionally, A can be applied to the right hand expression in equation 22. Since constants are unaffected by operators,

$$A(E_n^{(1)}\psi_n^{(1)}) = E_n^{(1)}(A\,\psi_n^{(1)}). \tag{25}$$

Because of equation 22, equations 24 and 25 are equivalent. Equating these gives

$$H_2(A\,\psi_n^{(1)}) = E_n^{(1)}(A\,\psi_n^{(1)}). \tag{26}$$

The result is an equation relating wave functions and energies from $H_1$ to $H_2$. By comparing equations 23 and 26, A acting upon $\psi_n^{(1)}$ results in some wave function $\psi_m^{(2)}$ with energy $E_n^{(1)}=E_m^{(2)}$. That is, there exists *some* solution, $\psi_m^{(2)}$, to $H_2$ that has the same energy as the given $\psi_n^{(1)}$. In other words, $H_1$ and $H_2$ share energy spectra! That is,

$$E_m^{(2)} = E_n^{(1)} \tag{27}$$

for some m and n. Despite the fact that $V_1$ and $V_2$ may look completely different in form, their wave function solutions possess the same energy levels. This is a critical result and is at the heart

of the present analysis. But what is the precise nature of this relationship? Consider if there was some constant coefficient embedded in the wave function. Since constants are not affected by operators, equations 22-26 all hold regardless of coefficients to the wave function. This is why the wave functions must be normalized. Those that have taken a course in quantum mechanics should be familiar with the process of normalization. For those are not, the process involves integrating the absolute square of the wave function over all of space and setting the result equal to 1. This ensures that the particle in question exists *somewhere* in space. A more detailed description of normalization can be found in Griffiths' text or in any undergraduate quantum mechanics text. Relating $\psi_n^{(1)}$ and $\psi_m^{(2)}$ starts with the following relation that is taken by comparing equations 23 and 26:

$$\psi_m^{(2)} = C(A\psi_n^{(1)}). \tag{28}$$

C is some normalization constant that is taken to be real. To calculate C, $\psi_m^{(2)}$ must be normalized. Note that it is assumed in equation 28 that $\psi_n^{(1)}$ is already normalized:

$$\int_{-\infty}^{\infty} \left|\psi_m^{(2)}(x)\right|^2 dx = \int \psi_m^{*(2)}\psi_m^{(2)} dx = 1. \tag{29}$$

The asterisk superscript represents the complex conjugate of the wave function and serves to eliminate any imaginary components from the square of the wave function. It also has an interesting effect on operators, which is highlighted in the jump from the first to the second line of equations below:

$$1 = \int \psi_m^{*(2)}\psi_m^{(2)} dx = \int (C(A\psi_n^{(1)}))^* (C(A\psi_n^{(1)}))dx = C^2 \int (A\psi_n^{(1)})^* (A\psi_n^{(1)})dx$$

$$= C^2 \int (\psi_n^{*(1)} A^\dagger)(A\psi_n^{(1)})dx = C^2 \int \psi_n^{*(1)} (A^\dagger A\psi_n^{(1)})dx = C^2 \int \psi_n^{*(1)} (E_n^{(1)}\psi_n^{(1)})dx$$

$$= C^2 E_n^{(1)} \int \psi_n^{*(1)} \psi_n^{(1)} dx = C^2 E_n^{(1)}$$

$$1 = C^2 E_n^{(1)}$$

$$\text{or} \quad C = \frac{1}{\sqrt{E_n^{(1)}}}. \tag{30}$$

The aforementioned "interesting effect" is the change from an operator to its 'Hermitian adjoint' or 'Hermitian conjugate'. The details of finding the conjugate on operator will not be gone over in this paper, but it suffices to say that $A^\dagger$ is the Hermitian conjugate of A. The switch in order of the operator and wave function is also the result of conjugation. For more information, see Griffiths. Notice that $A^\dagger$ and A are now both acting on $\psi_n^{(1)}$. Using equation 22, the operators can be replaced with $E_n^{(1)}$. Since the energy is just a constant, it can be taken to the outside of the integral which yields the final line of equation 26. But since $\psi_n^{(1)}$ is normalized, the integral term is equal to 1. What results is the final line of equation 26. Thus, the complete form of equation 28 is

$$\psi_m^{(2)} = \frac{1}{\sqrt{E_n^{(1)}}} A \psi_n^{(1)}. \tag{31}$$

An identical process can be performed to solve $\psi_n^{(1)}$ in terms of $\psi_m^{(2)}$, though it will not be explicitly shown here:

$$\psi_n^{(1)} = \frac{1}{\sqrt{E_m^{(2)}}} A^\dagger \psi_m^{(2)}. \tag{32}$$

Unfortunately, equations 31 and 32 are useless in their current forms because the relationship between m and n has not yet been established. This can be reasoned out with some critical

thinking. Recall that in solving for the superpotential, equation 15 was set to 0 since this would yield a solution to the SE:

$$A\psi_0^{(1)} = 0. \tag{33}$$

In quantum mechanics literature, one would say that the operator A annihilates the ground state wave function. This means that there is *no* $\psi_m^{(2)}$ with an energy equal to 0. The reason for this stems from the fact that equation 5 is an eigenvalue equation. As mentioned earlier, the reader should have a basic understanding of these equations and should know that an important aspect of them is that the eigenvector – in this case, the wave function – cannot take on a value of 0. Because of this, the ground state energy of $H_1$ does not associate directly with the ground state of $H_2$. Thus, $m \neq n$. Furthermore, since there must be *some* $\psi_n^{(1)}$ associated with $\psi_0^{(2)}$, $m < n$. That is, $m = n - r$, where r is some positive integer. It turns out that the only acceptable value for r is 1. To see why, let $r = 2$. A rough sketch of both potentials and their wave functions is provided in figure 2 below.

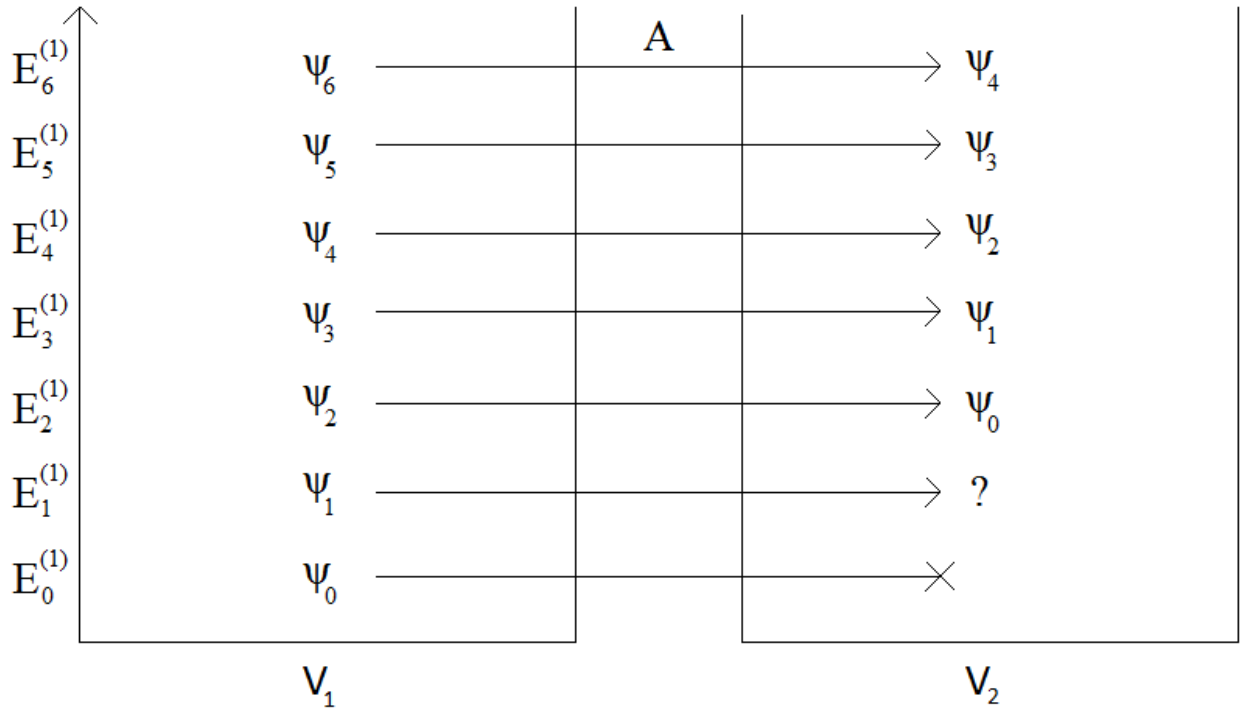



**Figure 2: Partner Potentials w/ Associated Wave Functions and Energies (m = n - 2)**

The arrows represent the operator A acting on $\psi_n^{(1)}$. As mentioned, $A\psi_0^{(1)}$ annihilates the wave function so $\psi_0^{(1)}$ does not have a $\psi_m^{(2)}$ associated with it. But what about $\psi_1^{(1)}$? According to equation 26, there exists some $\psi_m^{(2)}$ with energy $E_1^{(1)}$. But all of the $\psi^{(2)}$'s are already associated with $\psi^{(1)}$'s so $\psi_1^{(1)}$ has no wave function to associate with. Therefore, $m \neq n - 2$. (This argument hinges on the assumption that the potentials are reasonably well behaved so that $E_n^{(1)} > 0$ for all $n > 0$.) It is easy to see that how this argument extends to $r > 2$; there will always be some wave function that gets "skipped over" in $V_1$. More precisely, there will be exactly $r$ $\psi^{(1)}$'s that are skipped over. m and n must then be related in such a way that exactly one wave function gets skipped, $\psi_0^{(1)}$. And since $m > n$, the only solution is for $m = n - 1$. The complete relationship between n and m, and by extension $H_1$ and $H_2$, is now known. The information gathered thus far can be succinctly expressed with the following 4 equations:

$$E_0^{(1)} = 0, \tag{34}$$

$$E_n^{(2)} = E_{n+1}^{(1)}, \tag{35}$$

$$\psi_n^{(2)} = \frac{1}{\sqrt{E_{n+1}^{(1)}}} A \psi_{n+1}^{(1)}, \tag{36}$$

$$\psi_{n+1}^{(1)} = \frac{1}{\sqrt{E_n^{(2)}}} A^\dagger \psi_n^{(2)}. \tag{37}$$

Equations 34-37 represent the first major plateau in the analysis of supersymmetric quantum mechanics and will be the focus of the remainder of the paper. Notice that equation 34 is just the assumption that was made at the beginning of the analysis section. It *must* hold in order for the Hamiltonian to be factorized. Luckily, this condition can usually be forced on a system by subtracting off the ground state energy from all energy levels. It is important that one recognize that this scaling down of energy has no influence on the energy spacing or relationships between energy levels. Thus, generality is preserved. The rest of the equations allow one to switch between the two potentials through the A and $A^\dagger$ operators. A more formalized treatment would go on to show how this relationship represents the link between a particle and its supersymmetric partner. For now, it is merely a mathematical curiosity.

An application of the above treatment to a sample potential is appropriate at this point. In true QM fashion, the infinite square well is an excellent starting potential to try out this analysis. Equation 6 will play the role of $V_1$ and $H_1$ is defined in equation 12. The energy states of the infinite square well are given by equation 9. Since the ground state energy must be 0, $E_0$ must be subtracted off from all energy levels. The energy spectrum now has the form of equation 38:

$$E_n = n(n+2)\frac{\hbar^2 \pi^2}{2mL^2}. \tag{38}$$

Note that $(n+1)^2 - 1 = n(n+2)$. The wave function spectrum remains unchanged. Because the ground state wave function of $H_1$ is known, W(x) can be calculated using equation 19. The result is given in equation 39:

$$W(x) = -\frac{\hbar\pi}{L\sqrt{2m}}\cot\left(\frac{\pi x}{L}\right). \tag{39}$$

This can be used with equation 19 to find the supersymmetric partner potential to the infinite square well:

$$V_2(x) = \frac{\hbar^2\pi^2}{2mL^2}\left[2\text{cosec}^2\left(\frac{\pi x}{L}\right) - 1\right]. \tag{40}$$

Compare equation 40 with equation 6. Notice the striking dissimilarity between the potentials' functional forms. If presented with $V_2$ as a starting potential, it would prove difficult to find the analytical solution wave functions and their corresponding energy levels. SUSY allows one to solve this problem since it is the supersymmetric partner potential to a previously solved potential. Equation 36 allows one to calculate the solution wave functions exactly. The first few are provided below:

$$\psi_0^{(2)} = -\sqrt{\frac{2}{L}}\sin^2\left(\frac{\pi x}{L}\right), \tag{41}$$

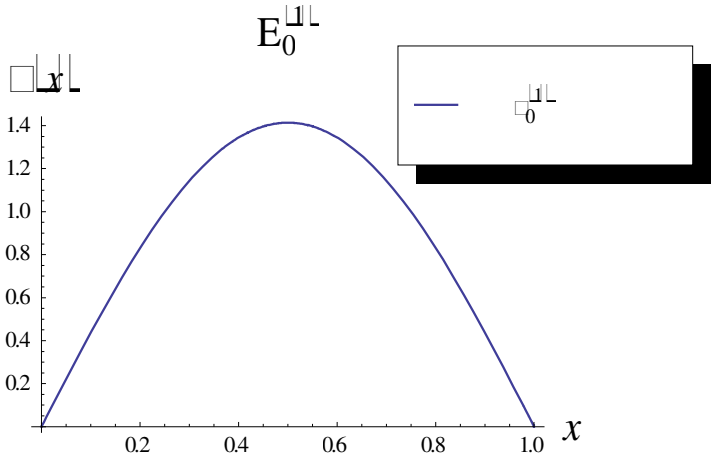
$$\psi_1^{(2)} = -\frac{4}{3}\sqrt{\frac{2}{L}}\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{2\pi x}{L}\right), \tag{42}$$

$$\psi_2^{(2)} = -\sqrt{\frac{2}{L}}\sin^2\left(\frac{\pi x}{L}\right)\left(2 + 3\cos\left(\frac{2\pi x}{L}\right)\right), \tag{43}$$

$$\psi_3^{(2)} = -\frac{8}{5}\sqrt{\frac{2}{L}}\sin^2\left(\frac{\pi x}{L}\right)\left(3\cos\left(\frac{\pi x}{L}\right)+2\cos\left(\frac{3\pi x}{L}\right)\right), \tag{44}$$

$$\psi_4^{(2)} = -\frac{1}{3\sqrt{2}}\sqrt{\frac{1}{L}}\left(6\cos\left(\frac{6\pi x}{L}\right)-\cot\left(\frac{\pi x}{L}\right)\sin\left(\frac{6\pi x}{L}\right)\right). \tag{45}$$

Note that equation 45 has not been simplified much, but there certainly exists some form similar to the preceding equations. $\psi_n^{(2)}$ increases in complexity at a substantial rate and would be difficult to calculate if not for SUSY QM despite the apparent dissimilarity between the two potentials in question. To further illustrate the relationship between partner wave functions, plots have been provided below in Figures 3 through 8. In each figure, corresponding wave functions with equal energies have been plotted together. For simplicity, $\hbar=\pi=2m=L=1$. Note that figure 3 has only 1 plot since $\psi_0^{(1)}$ has no partner wave function.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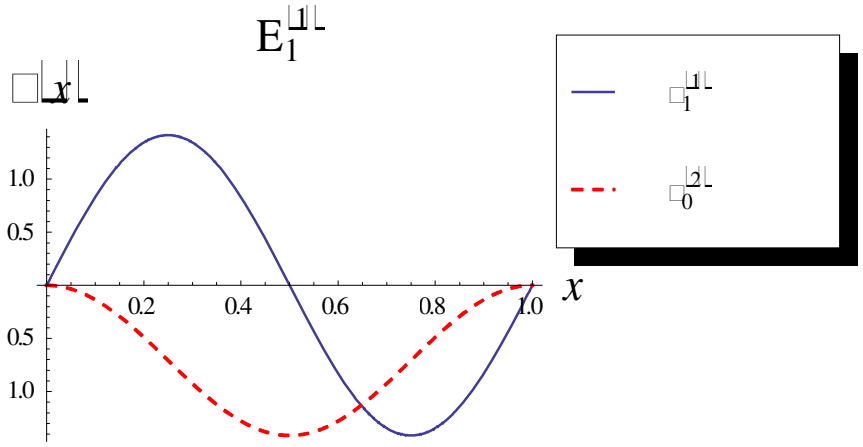
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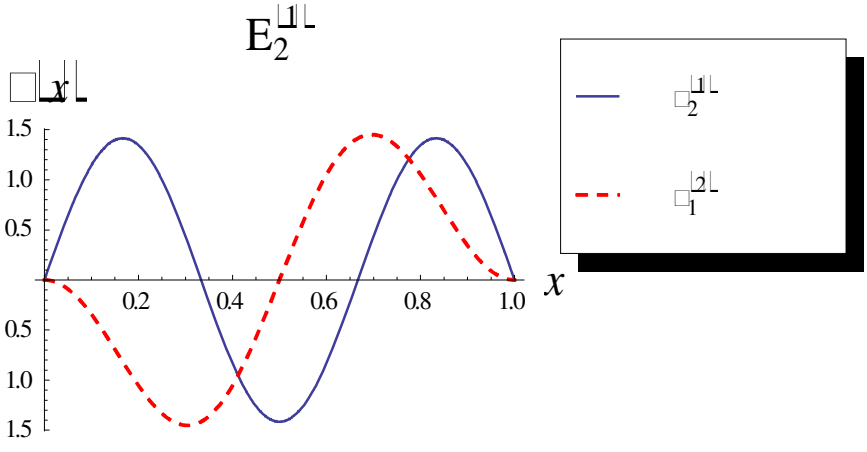
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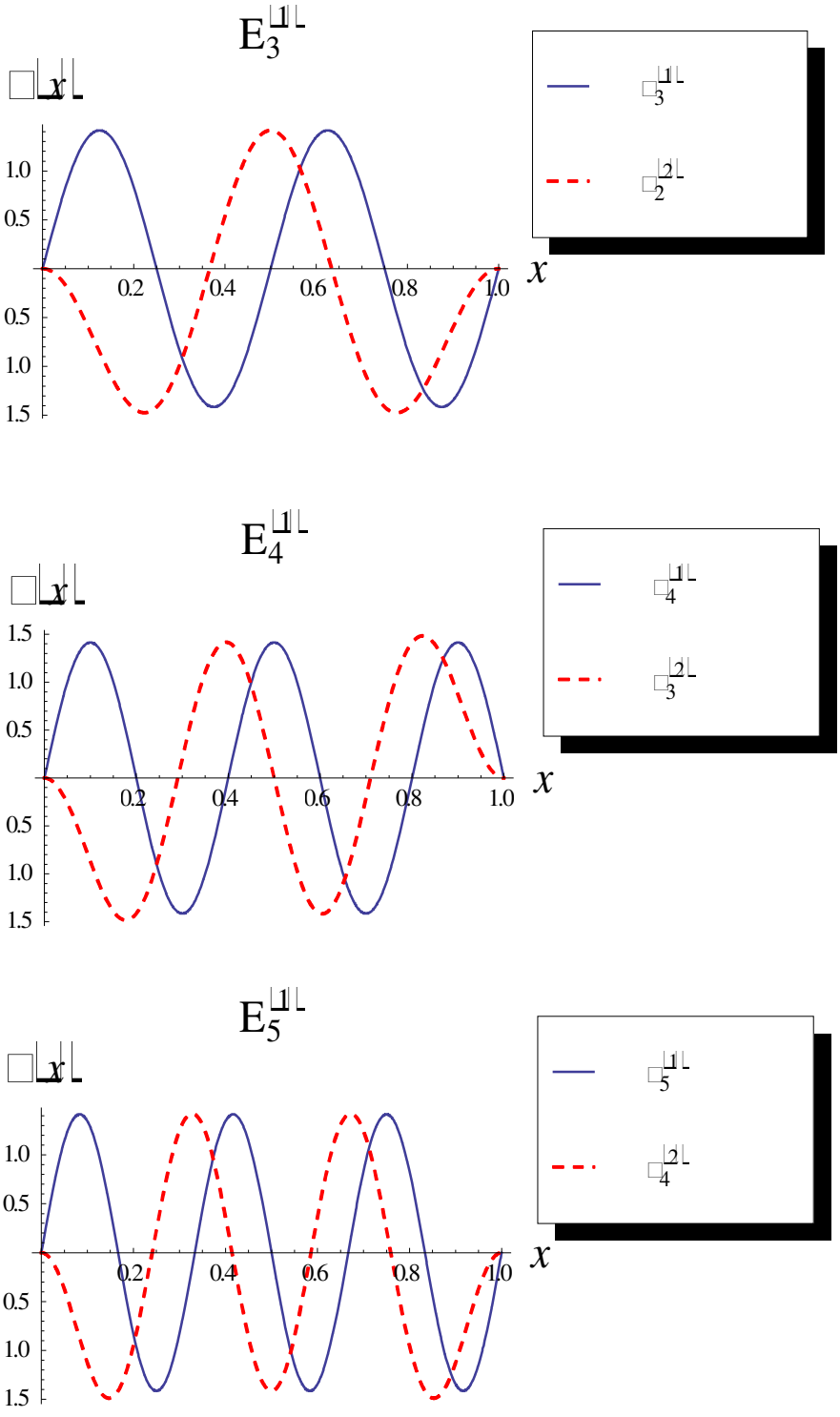


**Figures 3-8: Wave Function Solutions to the Infinite Square Well (blue/solid)**

**Supersymmetric Partner Potentials (red/dashed) (see Eqns. 41-45)**

It appears that in the space between the edges of the well, the partner functions are almost identical in shape and differ only in phase. At the edges of the well, both functions must go to 0 per the boundary conditions. Also notice that in each graph, $\psi^{(1)}$ has one more node than $\psi^{(2)}$. This is one of the results of A operating on $\psi^{(1)}$. In figure 3, $\psi^{(1)}$ only has one node so its partner function would have to have none i.e. it must be linear. The only way a linear function can satisfy the boundary conditions would be if it were the zero function. This is the graphical manifestation of equation 18, that is A annihilating $\psi_0^{(1)}$.

# *III. Conclusion*

It has been shown that with a clever factoring of the Hamiltonian into the A and $A^\dagger$ operators, a given potential $V_1(x)$ has an associated supersymmetric partner potential $V_2(x)$. The functional forms of $V_1$ and $V_2$ can differ dramatically, thus the relationship between these two potentials is highly non-trivial and requires the introduction of a new potential W(x) called the superpotential. Remarkably, if the wave function solutions to $V_1$ are known, then the function solutions to $V_2$ can be calculated directly with equation 31. Each solution in $V_1$ (excluding the ground state) has a partner solution in $V_2$. Furthermore, partner wave functions have the exact same energies so the energy spectrum for $V_2$ is already known from $V_1$. As mentioned at the beginning of the paper, a direct link between the above work and the prediction of supersymmetric partner particles will not be provided here. Nonetheless, the mathematical foundation for supersymmetric quantum mechanics has been provided at a level that any undergraduate physics student should be able to comprehend and is a mathematical curiosity that is worth exploring.

# *IV. References*


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